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GENERALIZATION OF EXERCISE 97.

By Marcus Baker, Washington, D. C.

Problem.—In the triangle ABC two lines drawn from C intersect the side AB. Given the angle C, the angle between the lines, and the ratios between the segments of AB; to find the remaining angles of the figure.

Solution.—1. Let M, M' be the points of intersection, such that

$$AM : M'B : AB = m : m' : 1.$$

Draw parallels to BC, intersecting AC at N and N'; whence

$$\frac{M'N'}{MN} \cdot \frac{CN}{CN'} = \frac{1-m'}{m} \cdot \frac{1-m}{m'}.$$

From the triangles CMN and $CM'N'$ we have

$$\frac{MN}{CN} = \frac{\sin NCM}{\sin CMN} = \frac{\sin ACM}{\sin MCB},$$

and
$$\frac{M'N'}{CN'} = \frac{\sin N'CM'}{\sin CM'N'} = \frac{\sin ACM'}{\sin M'CB};$$

whence
$$\frac{\sin(\alpha+\varphi)}{\sin[C-(\alpha+\varphi)]} \cdot \frac{\sin(C-\varphi)}{\sin\varphi} = k, \tag{1}$$

where $\varphi = \angle ACM$, $\alpha = \angle MCM'$, and $k = \dfrac{(1-m)(1-m')}{mm'}$.

Clearing of fractions, dividing by $[\cos\alpha\cos C - k\cos(C-\alpha)]\cos^2\varphi$, and transposing, we have

$$\tan^2\varphi - \frac{(1-k)\sin(C-\alpha)}{\cos\alpha\cos C - k\cos(C-\alpha)}\tan\varphi = \frac{\sin\alpha\sin C}{\cos\alpha\cos C - k\cos(C-\alpha)}, \tag{2}$$

from which to find φ.

2. Equation (2) may also be found in the following manner:—

If we designate the sides, as usual, by a, b, c, and put $MC = x$, $M'C = y$, $M'CB = \varphi'$, whence $\varphi + \alpha + \varphi' = C$; then from a consideration of equivalent areas,

$$ab\sin C = \frac{xy\sin\alpha}{1-m-m'} = \frac{bx\sin\varphi}{m} = \frac{ay\sin\varphi'}{m'};$$

whence
$$\frac{abxy\sin C\sin\alpha}{1-m-m'} = \frac{abxy\sin\varphi\sin\varphi'}{mm'},$$

or
$$\sin\varphi\sin\varphi' = \frac{mm'}{1-m-m'}\sin C\sin\alpha. \tag{3}$$

Also, by (1) $\sin\varphi\sin\varphi' = \dfrac{mm'}{(1-m)(1-m')}\sin(\alpha+\varphi)\sin(C-\varphi)$, (4)

or combining (3) and (4),

$$\sin(\alpha+\varphi)\sin(C-\varphi) = \frac{(1-m)(1-m')}{1-m-m'}\sin C\sin\alpha\,; \qquad (5)$$

which by expansion and reduction becomes (2) as already found.

φ' can, of course, be found in a manner entirely analogous; or φ being known, it can be obtained from (3) and checked by the relation $\varphi' = C-(\alpha+\varphi)$.

3. To find A and B we have

$$\frac{\sin\varphi}{\sin A} = \frac{mc}{MC} \quad\text{and}\quad \frac{\sin(C-\varphi)}{\sin B} = \frac{(1-m)c}{MC}\,;$$

whence $\dfrac{\sin B}{\sin A} = \dfrac{\sin(C+A)}{\sin A} = \sin C\cot A + \cos C = \dfrac{m}{1-m}\cdot\dfrac{\sin(C-\varphi)}{\sin\varphi}$;

$$\therefore\ \tan A = \frac{\sin C}{\dfrac{m}{1-m}\cdot\dfrac{\sin(C-\varphi)}{\sin\varphi} - \cos C},$$

and similarly,

$$\tan B = \frac{\sin C}{\dfrac{1-m}{m}\cdot\dfrac{\sin\varphi}{\sin(C-\varphi)} - \cos C}. \qquad (6)$$

4. As φ is obtained by means of a quadratic equation, the problem may be solved by a geometric construction. The following is such a construction: —

Upon any arbitrary line AB construct a circular segment containing the angle C; upon a segment MM' of this line construct another circular segment containing the angle α; the intersections of these segments in C and C' determine two triangles, whose angles A and B satisfy the required conditions. The proof is obvious, as is also the fact that there are in general two solutions.

This solution is due to Mr. James Main, of Washington, D. C., who solves this problem by co-ordinate geometry,* deriving equations to the two circles above drawn.

5. In the special case of a right-angled triangle, $C = \frac{1}{2}\pi$, and equations (2) and (6) become

$$\tan^2\varphi - \frac{1-m-m'}{(1-m)(1-m')}\cot\alpha\tan\varphi = -\frac{mm'}{(1-m)(1-m')} \qquad (2')$$

and

$$\tan A = \frac{1-m}{m}\tan\varphi. \qquad (6')$$

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*[See ANNALS OF MATHEMATICS Vol. II. p. 142.—O. S.]

If we substitute in (2′) the value of tan φ from (6′) and reduce, we have

$$\tan^2 A - \frac{1-m-m'}{m(1-m')}\cot\alpha\tan A = -\frac{m'(1-m)}{m(1-m')},$$

from which we may obtain A directly from the data without calculating the auxiliary φ. Also (3), (5), and (1) become

$$\sin\varphi\sin\varphi' = \frac{mm'}{1-m-m'}\sin\alpha, \tag{3′}$$

$$\cos\varphi\cos\varphi' = \frac{(1-m)(1-m')}{1-m-m'}\sin\alpha, \tag{5′}$$

$$\tan\varphi\tan\varphi' = \frac{mm'}{(1-m)(1-m')}. \tag{1′}$$

6. (6′) may also be derived in the following manner: —

If M_0 be the middle point of AB, N_0 the middle point of AC, and O the intersection of CM and M_0N_0; then, as may be easily seen,

$$\frac{M_0N_0}{ON_0} = \frac{\tan A}{\tan\varphi} = \frac{M_0N_0}{MN}\,\frac{MN}{ON_0} = \frac{1-m}{m},$$

or

$$\tan A = \frac{1-m}{m}\tan\varphi,$$

and similarly

$$\tan B = \frac{1-m'}{m'}\tan\varphi'. \tag{6′}$$

7. If the middle point of MN' coincides with M_0, $m = m'$; whence

$$\sin\varphi\sin\varphi' = \frac{m^2}{1-2m}\sin\alpha, \tag{3″}$$

$$\cos\varphi\cos\varphi' = \frac{(1-m)^2}{1-2m}\sin\alpha, \tag{5″}$$

$$\tan\varphi\tan\varphi' = \frac{m^2}{(1-m)^2}. \tag{1″}$$

Also, since tan B = cot A, (6′) gives

$$\tan A + \cot A = \frac{2}{\sin 2A} = \frac{1-m}{m}(\tan\varphi + \tan\varphi').$$

But

$$\tan\varphi + \tan\varphi' = \frac{\sin(\varphi+\varphi')}{\cos\varphi\cos\varphi'} = \frac{1-m-m'}{(1-m)(1-m')}\cot\alpha;$$

whence

$$\sin 2A = \frac{2m(1-m')}{1-m-m'}\tan\alpha, \tag{8}$$

by means of which A may be calculated without the use of auxiliaries.